Assignment 9.

1. (a) By sketching a suitable pair of graphs, show that there is only one value of x that is a root of the following equation [2]

$$\frac{x}{3} + 2 = \mathrm{e}^{-x}.$$

(b) Verify, by calculation, that this root lies between -1 and 0. [2]

(c) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \ln 3 - \ln(x_n + 6)$$

converges, then it converges to the root of the equation given in part (a). [2]

(d) Use this iterative formula, with $x_1 = 0$, to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

2. The constant a, where $a > 1$, is such that $\displaystyle\int_1^a \left(\sqrt{x} + \frac{1}{x}\right) \mathrm{d}x = 10$.

(a) Find an equation satisfied by a, and show that it can be written in the form [5]

$$a = \left(16 - \tfrac{3}{2}\ln a\right)^{\frac{2}{3}}.$$

(b) Verify, by calculation, that the equation $a = \left(16 - \frac{3}{2}\ln a\right)^{\frac{2}{3}}$ has a root between 5 and 6. [2]

(c) Use the iterative formula

$$a_{n+1} = \left(16 - \tfrac{3}{2}\ln a_n\right)^{\frac{2}{3}},$$

with $a_1 = 5$, to calculate the value of a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

3. A curve has equation $f(x) = \sqrt{1 + 2x^3}$, for $-0.5 \leq x \leq 0.5$.

   (a) Use the binomial expansion to express $f(x)$ in ascending powers of x, up to and including the term in x^6. Hence prove that $f(x) \approx 1 + x^3$ when $|x|$ is sufficiently small. [3]

   (b) Use the trapezium rule with four intervals to estimate the value of

$$\int_{-0.5}^{0.5} \sqrt{1 + 2x^3}\, \mathrm{d}x,$$

   giving your answer correct to 2 decimal places. [3]

   (c) Sketch the graph of $y = 1 + x^3$, labeling the points where the curve meets the x- and y-axes. [2]

   (d) Now assume that the graph of $f(x)$ is similar to the graph of $y = 1 + x^3$, for $-0.5 \leq x \leq 0.5$. Explain, with reference to your graph in part (c), why the estimate in part (b) may be expected to give a good approximation to the true value of the integral in this case. [1]

4. A curve has equation $y = \sqrt{2x + 1}$, for $x \geq -\frac{1}{2}$.

   (a) Sketch the curve, labeling the points where the curve meets the x- and y-axes. [2]

   (b) Evaluate the area under the curve between the lines $x = 0$ and $x = 4$. [3]

   (c) Use the trapezium rule with four intervals to estimate the area in part (b), giving your answer correct to 2 decimal places. [3]

   (d) The estimate found in part (c) is denoted by E. Explain, without further calculation, whether another estimate found using the trapezium rule with eight intervals would be greater than E or less than E. [2]

Total mark of this assignment: 38.